# Week 2. 拓扑基础习题课

No. LJL

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## 第一周作业反馈

★1. $A \subseteq X$ 闭 $\Rightarrow$ $A \subseteq X$ 不是开集 ✗

注：既开又闭的集合 eg. $X := (0,1) \cup (2,3) \subseteq \mathbb{R}$

则 $A := (0,1) \subseteq X$ 既开又闭

更一般 $\left.\begin{array}{l} X = \bigsqcup_\alpha U_\alpha \\ U_\alpha \subseteq X \text{ 开} \end{array}\right\} \Rightarrow \forall \alpha,\ U_\alpha \subseteq X$ 既开又闭.

2. $A \subseteq X$ dense $\Rightarrow$ $A$ 是开集 ✗

注：存在不稠密开集 eg. $(0,1) \subseteq [0,1]$ 不稠密开集

ps. $\overline{(0,1)}_{[0,1]} = \overline{(0,1)}_{\mathbb{R}} \cap [0,1] = [0,1]$

eg. $\mathbb{R}$ 拓扑 $\tau = \{U \subseteq \mathbb{R} \mid 0 \in U \text{ or } U = \phi\}$. 是拓扑 ✓

则 $\{0\} \subseteq (\mathbb{R}, \tau)$ 稠密开集.（dense 性：$x \in \mathbb{R}$ 的任意邻域均与之交）

3. $d(B(x,\delta), A)$ 歧义 ps: 一般指 $\inf\limits_{\substack{y \in B(x,\delta) \\ a \in A}} d(y, a)$

要想 $d(-,A)\,(B(x,\delta))$，建议设 $f(x) = d(x,A)$

$f(B(x_0,\delta))$

4. 错误证明 $d(x,y): X \times X \longrightarrow \mathbb{R}$ 连续.

注：要考虑 $X \times X$ 积拓扑

**Prop 1.** 度量空间常见结论 度量等价；子；积；连续性；$\overline{A}$

(1). $X$ 上两度量 $d, \rho$ 分别诱导拓扑 $\tau_1, \tau_2$

$\exists c > 0,\ \forall x, y \in X,\ d(x,y) \le c\,\rho(x,y) \Rightarrow \forall x \in X, \forall r > 0,\ B_d(x, cr) \supseteq B_\rho(x, r) \Rightarrow \tau_1 \subseteq \tau_2$

Cor. $\underbrace{\exists c_1, c_2 > 0,\ \forall x, y \in X,\ c_1\rho(x,y) \le d(x,y) \le c_2\rho(x,y)}_{\text{此时称度量等价}} \Rightarrow \tau_1 = \tau_2$

(2). $(X, d)$, $A \subseteq X$ $\Rightarrow$ $(A, d|_{A\times A})$ 诱导的度量拓扑 $=$ $A$ 上子空间拓扑

(3) $(X_i, d_i)$, $i = 1, \cdots, n$.

则 $\prod X_i$ 上有等价度量 $\begin{cases} \mu(x,y) = \sum a_i d_i(x_i, y_i), & \text{其中 } a_i > 0. \\ \rho(x,y) = \max\{d_i(x_i, y_i)\} \\ d(x,y) = \sqrt{\sum d_i(x_i,y_i)^2} \end{cases}$

应用：$X_i := \mathbb{R}$

ps: 等学了泛函分析，有些结论可以推到范数上……

且它们诱导的拓扑 $=$ 积空间拓扑.

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(4). $(X,d)$, $A\subseteq X$ 则

① $d: X\times X \longrightarrow X$ 连续

② $d(-,A): X\longrightarrow X$ 连续

③ $x\in\bar{A} \iff d(x,A)=0$

$\iff \exists a_n\in A,\ a_n\longrightarrow x$

④ $x_n\to x \iff d(x_n,x)\longrightarrow 0$

$\tau_1$ $\tau_2$

Lem: $x_n\to x$, $f$连续 $\Rightarrow f(x_n)\to f(x)$.

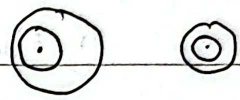
应用: $x_n\to x,\ y_n\to y \Rightarrow d(x_n,y_n)\to d(x,y)$

$x_n\to x \Rightarrow d(x_n,A)\to d(x,A)$

(5). 可数个度量空间的积空间可度量

但不可数个时,有反例 [Munkres, 21节, 例2]

如 $\sum_1^\infty \frac{1}{2^n}\frac{d_n(x_n,y_n)}{1+d_n(x_n,y_n)}$, $\sup_n\left\{\frac{\min\{d_n(x_n,y_n),1\}}{n}\right\}$



$\boxed{\text{Lem2}}$. $(X,d_1)$, $(X,d_2)$

$\forall x\in X,\ \forall r>0,\ \exists r'>0,\ B_{d_1}(x,r')\subseteq B_{d_2}(x,r) \Rightarrow \tau_1\supseteq\tau_2$

pf. $\forall B_{d_2}(x_0,r_0)$, $\forall x\in B_{d_2}(x_0,r_0)$ check 画图演示

在$\tau_2$中, $\exists r>0$, 使 $B_{d_2}(x,r)\subseteq B_{d_2}(x_0,r_0)$

由条件, $\exists r'>0$, $B_{d_1}(x,r')\subseteq B_{d_2}(x,r)\subseteq B_{d_2}(x_0,r_0)$.

$\therefore B_{d_2}(x_0,r_0)$ 是 $(X,d_1)$中开集

由拓扑基生成拓扑最小性 $\tau_2\subseteq\tau_1$

Back.

pf.(1) 由Lem2, 只需证:

$\exists c>0,\ \forall x,y\in X,\ d(x,y)\le c\rho(x,y) \Rightarrow \forall x\in X, r>0,\ B_d(x,cr)\supseteq B_\rho(x,r)$.

pf. $\forall x\in X,\ \forall r>0,\ \forall y\in B_\rho(x,r)$

$\therefore \cancel{d(x,y)\le r}\ \rho(x,y)\le r$ 定义验证.

$\therefore d(x,y)\le c\rho(x,y)\le cr$.

$\therefore y\in B_d(x,cr)$.

(2) $\boxed{\text{Lem3}}$. $(X,\tau)$有拓扑基$\mathcal{B}$, $A\subseteq X \Rightarrow \mathcal{B}\cap A$是$\tau\cap A$的拓扑基. Hint: 定义

pf. check $U\in\tau$, $\forall a\in A$, $\exists B\in\mathcal{B}$ $a\in B\subseteq U \Rightarrow a\in B\cap A\subseteq U\cap A$.

Back.

A上有拓扑基 $\{B(x,r)\cap A \mid x\in A, r>0\}$ $\Rightarrow$ 拓扑一样.

$(A, d|_{A\times A})$上诱导的拓扑的拓扑基

$$\begin{cases} d(x,y) \geq 0 \quad =0 \Leftrightarrow x=y \\ d(x,y)=d(y,x) \\ d(x,y) \leq d(x,z)+d(z,y) \end{cases}$$

(3). 先验证为度量.

$$\begin{cases} \mu(x,y) = \sum_1^n a_i d_i(x_i,y_i) \geq 0 \quad \begin{cases} =0 \Rightarrow \forall i, a_i d_i(x_i,y_i)=0 \text{ 即 } x_i=y_i \Rightarrow x=y \\ x=y \Rightarrow d_i(x_i,y_i)=0 \Rightarrow \mu(x,y)=0 \end{cases} \\ \mu(x,y) = \sum_1^n a_i d_i(x_i,y_i) = \sum_1^n a_i d_i(y_i,x_i) = \mu(y,x) \\ \mu(x,y) = \sum_1^n a_i d_i(x_i,y_i) \leq \sum_1^n a_i \left(d_i(x_i,z_i)+d_i(z_i,y_i)\right) = \mu(x,z)+\mu(z,y) \end{cases}$$

$$\begin{cases} \rho(x,y) = \max\{d_i(x_i,y_i)\} \geq 0 \quad \begin{cases} =0 \Rightarrow \forall i,\ d_i(x_i,y_i)=0 \Rightarrow \forall i,\ x_i=y_i \Rightarrow x=y \\ x=y \Rightarrow 0=0 \end{cases} \\ \rho(x,y) = \max\{d_i(x_i,y_i)\} = \max\{d_i(y_i,x_i)\} = \rho(y,x) \\ \text{设 } \rho(x,y) = \max\{d_i(x_i,y_i)\} \stackrel{\text{记}}{=} d_j(x_j,y_j) \leq d_j(x_j,z_j)+d_j(z_j,y_j) \\ \qquad \leq \max\{d_i(x_i,z_i)\} + \max\{d_i(z_i,y_i)\} \\ \qquad = \rho(x,z)+\rho(z,y) \end{cases}$$

$$\begin{cases} d(x,y) = \sqrt{\sum d_i(x_i,y_i)^2} \geq 0 \quad =0 \Leftrightarrow \sum d_i(x_i,y_i)^2=0 \Leftrightarrow x_i=y_i \Leftrightarrow x=y \\ d(x,y) = \sqrt{\sum d_i(x_i,y_i)^2} = \sqrt{\sum d_i(y_i,x_i)^2} = d(y,x) \\ \because \text{在 } \mathbb{R}^n \text{ 中 } \|a+b\| \leq \|a\|+\|b\| \\ \therefore d(x,y) = \sqrt{\sum d_i(x_i,y_i)^2} \leq \sqrt{\sum [d_i(x_i,z_i)+d_i(z_i,y_i)]^2} \\ \qquad \leq \sqrt{\sum d_i(x_i,z_i)^2} + \sqrt{\sum d_i(z_i,y_i)^2} \\ \qquad = d(x,z)+d(z,y) \end{cases}$$

再验证度量等价:

需要用 $\mathbb{R}$ or $\mathbb{R}^n$ 上一些不等式:

$$\sqrt{x_1^2+\cdots+x_n^2} \underset{\text{两平方证}}{\leq} |x_1|+\cdots+|x_n| \underset{\text{Cauchy}}{\leq} \sqrt{n}\sqrt{x_1^2+\cdots+x_n^2}$$

这说明 $a_i=1$ 时, $\mu$ 与 $d$ 度量等价

$\Rightarrow \forall a_i>0,\ b_i \geq 0$ 由 $a \cdot b \leq \|a\|\|b\|$

$$\min\{a_i\}\sqrt{b_1^2+\cdots+b_n^2} \leq \min\{a_i\}(b_1+\cdots+b_n) \leq \sum a_i b_i \leq \sqrt{a_1^2+\cdots+a_n^2}\sqrt{b_1^2+\cdots+b_n^2}$$

$$\therefore \min\{a_i\}\, d(x,y) \leq \min\{a_i\}\, s(x,y) \leq \mu(x,y) \leq \sqrt{a_1^2+\cdots+a_n^2}\, d(x,y), \quad s(x,y)=\sum_1^n |x_i-y_i|$$

$\therefore$ $\mu$ 与 $d$ 度量等价

$$\begin{cases} \sqrt{x_1^2+\cdots+x_n^2} \geq \max|x_i| \\ \sqrt{x_1^2+\cdots+x_n^2} \leq \sqrt{\max|x_i|^2+\cdots+\max|x_i|^2} \end{cases}$$

又 $\begin{cases} \rho(x,y) \leq d(x,y) \\ d(x,y) \leq \sqrt{n\,\rho(x,y)^2} = \sqrt{n}\,\rho(x,y) \end{cases}$

$\therefore$ $\rho$ 与 $d$ 度量等价

最后证明：度量$\rho$诱导的拓扑 = 积空间拓扑。

$\forall \prod X_i$ 中拓扑基中元素 $\prod_1^n B_{d_i}(x_i, r_i)$

$\forall x' \in \prod_1^n B_{d_i}(x_i, r_i)$，$x_i' \in B_{d_i}(x_i, r_i) \Rightarrow \exists r_i' > 0$，$B_{d_i}(x_i', r_i') \subseteq B_{d_i}(x_i, r_i)$

取 $r' := \min_{1\le i\le n}\{r_i'\}$ 则 $\forall i$，$B_{d_i}(x_i', r') \subseteq B_{d_i}(x_i, r_i)$

断言：$B_\rho(x', r') \subseteq \prod_1^n B_{d_i}(x_i, r_i)$

$\forall y \in B_\rho(x', r') \Rightarrow \forall i$，$d_i(x_i', y_i) < r'$

$\therefore y_i \in B_{d_i}(x_i', r') \subseteq B_{d_i}(x_i, r_i)$

$\therefore y \in \prod_1^n B_{d_i}(x_i, r_i)$.

$\forall B_\rho(x, r)$

$y \in B_\rho(x, r) \Leftrightarrow \forall i$，$d(x_i, y_i) < r$ 即 $y_i \in B_{d_i}(x_i, r)$

$\therefore B_\rho(x, r) = \prod_1^n B_{d_i}(x_i, r)$

(4). ①. $X \times X$ 上度量 $s((x_1, x_2), (y_1, y_2)) = d(x_1, y_1) + d(x_2, y_2)$

只需证：$|d(x_1, x_2) - d(y_1, y_2)| \le s((x_1, x_2), (y_1, y_2))$

$d(x_1, x_2) \le d(x_1, y_1) + d(y_1, y_2) + d(y_2, x_2)$

$\Rightarrow d(x_1, x_2) - d(y_1, y_2) \le d(x_1, y_1) + d(x_2, y_2)$

同理 $d(y_1, y_2) - d(x_1, x_2) \le d(x_1, y_1) + d(x_2, y_2)$

$\therefore |d(x_1, x_2) - d(y_1, y_2)| \le d(x_1, y_1) + d(x_2, y_2)$

②. $\forall x, y \in X$，$a \in A$，

$d(x, a) \le d(x, y) + d(y, a)$

$\Rightarrow d(x, A) \le d(x, y) + d(y, A)$

$\Rightarrow |d(x, A) - d(y, A)| \le d(x, y)$

③. $x \in \bar{A} \Rightarrow \forall B(x, \frac{1}{n}) \cap A \ne \emptyset \Rightarrow \exists x_n \in B(x, \frac{1}{n}) \cap A \Rightarrow x_n \to x$

若 $x_n \to x$，$x_n \in A \Rightarrow \forall U(x)$，$\exists N$，$\forall n \ge N$，$x_n \in U \cap A \Rightarrow U \cap A \ne \emptyset$

$x \in \bar{A} \Leftrightarrow \exists a_n \in A$，$a_n \to x \Leftrightarrow d(x, A) = 0$

"$\Rightarrow$"：$d(x, A) = \lim d(x_n, A) = 0$

"$\Leftarrow$"：$\exists a_n \in A$，$d(x, a_n) < \frac{1}{n}$

即 $a_n \to x$.

④ 如 $x_n \to x$

$\Leftrightarrow \forall \varepsilon > 0$，$\exists N_{\in\mathbb{N}}$，$\forall n \ge N$，$x_n \in B(x, \varepsilon)$ $\Leftrightarrow d(x, x_n) < \varepsilon$

$\Leftrightarrow d(x_n, x) \to 0$.

证明是闭集
- $\bar{A} \subseteq A$ △
- $A' \subseteq A$
- $A^c$ 是开集 △
- $A = f^{-1}(B)$，f 连续，B 闭 △



P26. 1. 证明：$X: T_2 \Leftrightarrow \Delta(X) \subseteq X \times X$ 闭

pf. $\Delta(X) \subseteq X \times X$ 闭 $\Leftrightarrow \Delta_X^c$ 开

$\Leftrightarrow \underbrace{\forall (x,y) \in \Delta_X^c}_{\Updownarrow \; x \neq y \text{ in } X}, \exists U(x) \subseteq X$ 开, $V(y) \subseteq X$ 开, $U \times V \subseteq \Delta_X^c$

$U \times V \subseteq \Delta_X^c \Leftrightarrow (U \times V) \cap \Delta_X = \phi \Leftrightarrow U \cap V = \phi$

$x \in U \cap V \Leftrightarrow (x,x) \in \Delta_X \cap (U \times V)$

$\Leftrightarrow \forall x \neq y$ in $X$, $\exists U(x)$ 开, $V(y)$ 开, $U \cap V = \phi$

$\Leftrightarrow X: T_2$ 空间

2. 证明：前面写过了（重叠）

3. 证明：$X, Y: T_2$ 则 $X \times Y: T_2$

pf. $\forall (x,y) \neq (x',y')$，

不妨设 $x \neq x'$ 　　　　用定义

由 $X: T_2 \Rightarrow \exists U(x)$ 开, $V(x')$ 开, $U \cap V = \phi$

$\therefore \begin{cases} U \times Y, V \times Y \text{ 分别为 } (x,y), (x',y') \text{ 开邻域} \\ (U \times Y) \cap (V \times Y) = (U \cap V) \times Y = \phi \end{cases}$

$\therefore X \times Y: T_2$

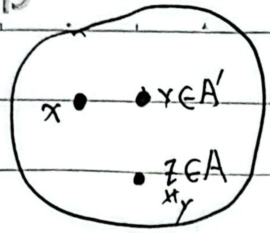
4. 证明：$T_1$ 空间的任一子集的导集是闭集

pf. 设 $X: T_1$ 空间，$A \subseteq X$

回顾 $\begin{cases} x \in A' \Leftrightarrow \forall O(x) \text{开}, (O - \{x\}) \cap A \neq \phi \\ x \in \bar{A} \Leftrightarrow \forall O(x) \text{开}, O \cap A \neq \phi \end{cases}$

法一：$A' \subseteq X$ 闭 $\Leftrightarrow \overline{A'} \subseteq A'$

$\Leftrightarrow \begin{pmatrix} \forall x \in \overline{A'} : \forall O(x) \text{开}, O \cap A' \neq \phi \\ \text{则 } x \in A' : \forall O(x) \text{开}, (O - \{x\}) \cap A \neq \phi \end{pmatrix}$




$x\in\overline{A'}$ ⟺: $\forall O(x),\ \exists y\in O\cap A'$

$x\in\overline{A'}$　若 $\exists O(x)$开, $\exists\ x\in O\cap A'$　✓

不妨设 $\forall O(x)$开, $x\notin O\cap A'$

由于 $O\cap A'\neq\phi$　取 $y\in O\cap A'$　则 $y\neq x$

由于 $X: T_1$,　$O-\{x\}$　是 $y$ 的开邻域.

由 $y\in A' \Rightarrow (O-\{x,y\})\cap A\neq\phi$, 取 $z$

$\therefore z\in(O-\{x\})\cap A$　从而非 $\phi$

$\therefore x\in A'$

~~法一: $X=A\sqcup(A')^c = B\sqcup A'\sqcup(A')^c$, B为A的孤立点集~~

~~$\therefore A'\subseteq X$闭 ⟺ $B\sqcup(A')^c$开~~

~~只需证 $B\subseteq X$开~~

法二: $\overline{X}=B\sqcup A'$, B为A的孤立点集

$A'\subseteq X$闭 ⟺ $A'\subseteq\overline{X}$闭 ⟺ $B\subseteq\overline{X}$开.

$\forall x\in B$, $\exists O\subseteq X$开,　$O\cap A=\{x\}$.

$O\cap\overline{A}=\{x\}$?　假设 $\exists y\in O\cap\overline{A}$ 且 $y\neq x$

$\therefore$ ~~$y\in A$~~, $O-\{x\}$ 则是 $y$ 开邻域

但 $(O-\{x\})\cap A=\phi$　矛盾于 $y\in\overline{A}$

$\therefore O\cap\overline{A}=\{x\}\Rightarrow\{x\}\subseteq\overline{A}$开.

$\therefore B\subseteq\overline{X}$开.

梳理

$T_1$ ⟺ 单点集闭

⇑

$T_2$ ⟺ $\Delta_X$闭 ⇒ 序列极限点唯一

⇑ $T_1$

$T_3$ ⟺ $\forall x\in U$, $U$开, $\exists V(x)$开, $x\in V\subseteq\overline{V}\subseteq U$

⇑ $T_1$

度量空间 ⟹ $T_4$ ⟺ $\forall A\subseteq U$ $\begin{cases}A闭\\U开\end{cases}$, $\exists V$开, $A\subseteq V\subseteq\overline{V}\subseteq U$

⇓　⟺ Uryshon引理: 连续函数分离两无交闭集.

Tietze扩张定理: 闭子空间上延拓

$$\begin{tikzcd} A \arrow[r] \arrow[d] & {[0,1]} \\ X \arrow[ur, dashed] & \end{tikzcd}$$

5. $(X,T):T_2$, $f:X\to X$ 连续 证明：不动点集 $Fix(f)\subseteq X$ 闭

pf. $(X,T):T_2 \Rightarrow \Delta_X\subseteq X\times X$ 闭.

$F: X\longrightarrow X\times X$, $x\longmapsto (f(x),x)$ 连续

则 $F^{-1}(\Delta_X)=Fix(f)$

$(x,x)\in$ 左边 $\iff f(x)=x \iff x\in$ 右边

$\therefore Fix(f)=F^{-1}(\Delta_X)$ 闭

推广: $f,g: X\longrightarrow Y$ 连续, $Y:T_2$ 则 $\{x\in X\mid f(x)=g(x)\}\subseteq X$ 闭.

pf. $F: X\longrightarrow Y\times Y$, $x\longmapsto (f(x),g(x))$ 连续.

$\{x\mid f(x)=g(x)\}=F^{-1}(\Delta_X)$ 闭.

6. $(Y,T):T_2$, $f:X\to Y$ 连续, 证明: $G_f:=\{(x,f(x))\mid x\in X\}\subseteq X\times Y$ 闭

pf. Hint: $(x,y)\in G_f \iff y=f(x)$

$F: X\times Y\longrightarrow Y\times Y$ 连续. $\Leftarrow \begin{cases} X\times Y\xrightarrow{p_1}X\xrightarrow{f}Y \\ X\times Y\xrightarrow{p_2}Y \end{cases}$

$(x,y)\longmapsto (f(x),y)$

断言: $F^{-1}(\Delta_Y)=G_f$

$(x,y)\in G_f \iff y=f(x) \iff (x,y)\in F^{-1}(\Delta_Y)$

$\therefore G_f=F^{-1}(\Delta_Y)$ 闭.

7. $(X,T):T_3$, $A\subseteq X$ 闭, $x\notin A$ 证明: $\exists\ U(x)$开, $V(A)$开, $\bar U\cap\bar V=\emptyset$.

pf. 由 $T_3\to \exists\ U_0(x)$开, $V_0(A)$开 $U_0\cap V_0=\emptyset$

对于 $x\in U_0 \xRightarrow{T_3} \exists\ U(x)$开, $x\in U\subseteq\bar U\subseteq U_0$

取 $V=V_0$

$\bar U\cap \bar V=\emptyset \Leftarrow \bar U\subseteq \bar V^c=(V^c)^\circ$

而: $\bar U\subseteq U_0\subseteq V^c \Rightarrow \bar U\subseteq U_0\subseteq (V^c)^\circ$

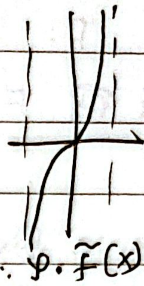
8. $(X,T)$: $T_4$, $A\subseteq X$ 闭, $f: A\to\mathbb{R}^n$ 连续

证明: $\exists$ $F: X\to\mathbb{R}^n$ 连续, $F|_A = f$.

pf. 先证 $n=1$ 时成立.

$\boxed{\text{Lem}}$. $\mathbb{R}\cong(-1,1)$

$$\begin{cases} g: (-1,1)\longrightarrow\mathbb{R}, & t\longrightarrow \tan\left(\frac{\pi}{2}t\right) \\ h: \mathbb{R}\longrightarrow(-1,1), & y\longrightarrow \frac{2}{\pi}\arctan t \end{cases}$$



Back.

对于 $A\xrightarrow{f}\mathbb{R}\xrightarrow[\cong]{h}(-1,1)\hookrightarrow[-1,1]$

$\exists$ $\tilde{f}: X\longrightarrow[-1,1]$, $\tilde{f}|_A = h\circ f$

$B=\tilde{f}^{-1}(\{\pm1\})$ 与 $A$ 无交闭集

∴ 由 Urysohn 引理, $\exists$ $\varphi: X\to[0,1]$ 连续, $\varphi|_B=0$, $\varphi|_A=1$.

∴ $\varphi\cdot\tilde{f}(x)\in(-1,1)$.

考虑: $h^{-1}\circ(\varphi\cdot\tilde{f})$

则可验证它是 $f$ 的连续延拓.

再证一般情况

设 $f: A\longrightarrow\mathbb{R}^n$, $a\longrightarrow(f_1(a),\cdots,f_n(a))$ 则 $f_i=\pi_i\circ f$ 连续

$\forall i$,
$$\begin{tikzcd} A \arrow[r, "f_i"] \arrow[d, hook] \arrow[r, phantom, shift right=3, "\Downarrow"] & \mathbb{R} \\ X \arrow[ur, "F_i"'] & \end{tikzcd}$$

def: $F: X\longrightarrow\mathbb{R}^n$, $a\longrightarrow(F_1(a),\cdots,F_n(a))$ 则 $\begin{cases} F\text{ 连续} \\ F|_A=f \end{cases}$

9. 举例说明 Tietze 扩张定理中, $A\subseteq X$ 闭, 条件不可去掉.

Hint: $f: \mathbb{R}-\{0\}\to[0,1]$

pf. $f: \mathbb{R}-\{0\}\longrightarrow[0,1]$, $\begin{cases} 1, & x>0 \\ 0, & x<0 \end{cases}$

~~f 连续 (0,∞)~~

∴ $f$ 连续 by the pasting lemma.

但 不可能 $\exists$ $g: \mathbb{R}\to[0,1]$, $g|_{\mathbb{R}-\{0\}}=f$, $g$ 连续. 数学分析, $\varepsilon$-$\delta$ 语言

补充: 通过考虑 拓扑基: 子空间与积空间拓扑

$X_i\longrightarrow A_i\subseteq X_i\longrightarrow\prod A_i$

$X_i\searrow\prod X_i\longrightarrow\prod A_i$

两种拓扑一样

补充：$(X_i, d_i)$ 度量空间 $\Rightarrow \prod X_i$ 可度量 的证明.

$\boxed{\text{Lem.}}$ 对于度量空间 $(X,d)$，$\rho(x,y) := \dfrac{d(x,y)}{1+d(x,y)}$ （或者 $\min\{d(x,y), 1\}$）

则 $\rho$ 是度量 且度量拓扑与原来一样

pf. 对于 $\rho(x,y) = \dfrac{d(x,y)}{1+d(x,y)}$
$$\begin{cases} \rho(x,y) \geq 0 \\ \rho(x,y) = 0 \Leftrightarrow d(x,y) = 0 \Leftrightarrow x = y \\ \rho(x,y) = \rho(y,x) \end{cases}$$

Lem: $a, b \geq 0$ 则 $\dfrac{a+b}{1+a+b} \leq \dfrac{a}{1+a} + \dfrac{b}{1+b}$

$\dfrac{a+b}{1+a+b} = \dfrac{a}{1+a+b} + \dfrac{b}{1+a+b} \leq \dfrac{a}{1+a} + \dfrac{b}{1+b}.$

$\therefore \rho$ 是度量，再证拓扑一样

对于 $B_d(x,r)$

$\dfrac{a}{1-a} \to 0 \ (a \to 0^+)$

$\dfrac{a}{1+a} < b \Leftrightarrow a < \dfrac{b}{1-b}$

$\therefore \exists r' \in (0,1)$，$\dfrac{r'}{1-r'} < \min\{\frac{1}{2}, r\}$

断言：$B_\rho(x, r') \subseteq B_d(x, r)$

$\forall y \in B_\rho(x, r')$，$\rho(x,y) < r' \Rightarrow d(x,y) < \dfrac{r'}{1-r'} < r.$

$f(d)$ 可推广为

① $f(0) = 0$

② $f\uparrow$；$\exists \varepsilon > 0$，$[0,\varepsilon]$ 严格↑ 连续

③ $f(a+b) \leq f(a) + f(b)$

对于 $B_\rho(x, r)$

$\dfrac{a}{1+a} \uparrow \ (a > 0)$

断言：$B_d(x, \dfrac{r}{1+r}) \subseteq B_\rho(x, r)$.

对于 $\rho(x,y) = \min\{d(x,y), 1\}$
$$\begin{cases} \rho(x,y) \geq 0, \\ \rho(x,y) = 0 \Leftrightarrow d(x,y) = 0 \Leftrightarrow x = y \\ \rho(x,y) = \rho(y,x) \end{cases}$$

$\forall x,y,z\in X$,

若 $d(x,z)>1$ or $d(z,y)>1$ 则

$$\rho(x,y)\le 1\le \rho(x,z)+\rho(z,y)$$

若 $d(x,z)\le 1$ 且 $d(z,y)\le 1$ 则

$$\rho(x,y)\le d(x,y)\le d(x,z)+d(z,y)=\rho(x,z)+\rho(z,y).$$

$\therefore$ $\rho$ 是个度量，下证拓扑一样

考虑半径 $<1$ 的开球作为拓扑基，则它们拓扑基一样.

例. $(X_i,d_i)$ 度量空间，$\forall i$, $\forall x,y\in X_i$, $d_i(x,y)\le 1$

则 在 $\prod X_i$ 上取 $\rho(x,y):=\sup\{\frac{d_n(x_n,y_n)}{n}\}$，$\mu(x,y)=\sum_{n=1}^{\infty}\frac{d_n(x_n,y_n)}{2^n}$

则 $\mu,\rho$ 均是度量 且拓扑 = 乘积拓扑

pf: 对于 $\rho(x,y)=\sup_n\{\frac{d_n(x_n,y_n)}{n}\}$.

$$\begin{cases}\rho(x,y)\ge 0,\quad =0 \Leftrightarrow \forall n,\ \frac{d_n(x_n,y_n)}{n}=0 \Leftrightarrow x=y\\ \rho(x,y)=\rho(y,x)\\ \forall x,y,z\in\prod_i X_i,\quad \forall n,\quad \frac{d_n(x_n,y_n)}{n}\le \frac{d_n(x_n,z_n)}{n}+\frac{d_n(z_n,y_n)}{n}\end{cases}$$

$$\le \rho(x,z)+\rho(z,y)$$

$$\therefore\ \rho(x,y)\le\rho(x,z)+\rho(z,y).$$

$\therefore$ $\rho$ 是度量. 下证度量拓扑 = 积拓扑

对于 $\prod U_i$，$U_i\subseteq X_i$ 开；$i>n$ 时 $U_i=X_i$

$\forall (x_i)\in\prod U_i$，则 $\forall 1\le i\le n$，$\exists r>0$，$B_{d_i}(x_i,r)\subseteq U_i$

断言：$B_\rho(x,\frac{r}{n})\subseteq \prod U_i$

$\forall y\in B_\rho(x,\frac{r}{n}) \Rightarrow \forall 1\le i\le n$，$\frac{d_i(x_i,y_i)}{i}<\frac{r}{n}$

$\therefore d(x_i,y_i)<\frac{ir}{n}\le r$

$\therefore y_i\in B_{d_i}(x_i,r)\subseteq U_i$，$1\le i\le n$

$\therefore y\in\prod U_i$

对于 $B_\rho(x,r)$

$\exists n\in\mathbb{N}_+$，$(n+1)r>2$

断言：$\prod U_i\subseteq B_\rho(x,\frac{r}{2})$，其中 $U_i=\begin{cases}B_{d_i}(x_i,\frac{r}{2}), & 1\le i\le n\\ X_i, & i>n\end{cases}$

$\forall (y_i)\in\prod U_i$，则 $\begin{cases}\forall 1\le i\le n, & \frac{d(x_i,y_i)}{i}<\frac{r}{2i}\le\frac{r}{2}.\\ \forall i>n, & \frac{d(x_i,y_i)}{i}\le\frac{1}{i}<\frac{r}{2}\end{cases}\Rightarrow y\in B_\rho(x,r).$

对于 $\mu(x,y) = \sum_{1}^{\infty} \frac{d_n(x_n, y_n)}{2^n}$

$$\begin{cases} \mu(x,y) \geq 0, \quad \mu(x,y)=0 \Leftrightarrow x=y \\ \mu(x,y)=\mu(y,x) \\ \text{三角不等式成立} \quad (\text{Hint: 级数和}) \end{cases}$$

$\therefore \mu$ 是度量.

$\forall \prod U_i$, $U_i \subseteq X_i$ 开; $\forall i > n$, $U_i = X_i$

$\forall x \in \prod U_i$, $\forall 1 \leq i \leq n$, $\exists r$, s.t. $B_{d_i}(x_i, r) \subseteq U_i$.

断言: $B_\mu(x, \frac{r}{2^n}) \subseteq \prod U_i$

$\forall y \in B_\mu(x, \frac{r}{2^n})$ 则 $\forall 1 \leq i \leq n$, $\frac{d_i(x_i, y_i)}{2^i} < \frac{r}{2^n}$

$\therefore y_i \in B_{d_i}(x_i, r)$ $(1 \leq i \leq n)$

$\therefore y \in \prod U_i$

$\forall B_\mu(x, r)$.

$\exists n \in \mathbb{N}_+$, s.t. $\frac{1}{2^{n+1}} + \frac{1}{2^{n+2}} + \cdots < \frac{r}{2}$

取 $U_i := \begin{cases} B_{d_i}(x, \frac{r}{2}), & 1 \leq i \leq n \\ X_i & i > n \end{cases}$

断言: $\prod U_i \subseteq B_\mu(x, r)$.

$\forall y \in \prod U_i$ 则 $\forall 1 \leq i \leq n$, $d_i(x_i, y_i) < \frac{r}{2}$

$\therefore \mu(x,y) = \left(\frac{d_1(x_1,y_1)}{2} + \cdots + \frac{d_n(x_n,y_n)}{2^n}\right) + \sum_{i=n+1}^{\infty} \frac{d_i(x_i,y_i)}{2^i}$

$\leq \frac{r}{2} \cdot \left(\frac{1}{2} + \cdots + \frac{1}{2^n}\right) + \sum_{i=n+1}^{\infty} \frac{1}{2^i}$

$< \frac{r}{2} + \frac{r}{2} = r$

$\therefore y \in B_\mu(x, r)$.

结合 Lem + prop, 即可证明开始命题

No.

Date / /